Steady-State, Turnover Rate, Residence Time and Stability for Donor-Controlled Linear Ecosystem Models

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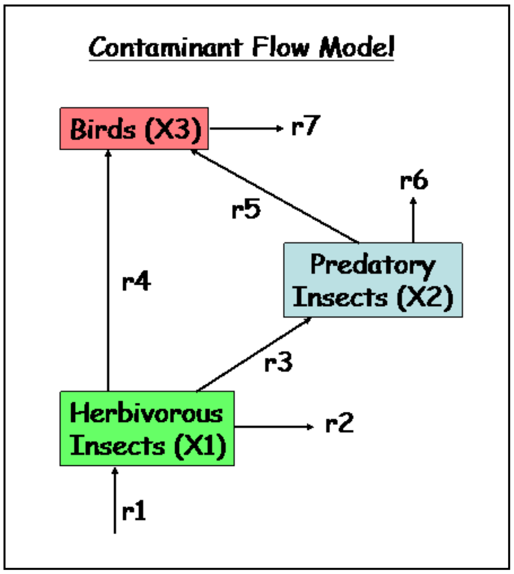
Donor-controlled linear ecosystem models are models in which each flux between compartments, or from a compartment to the "outside" of the system, is equal to a constant times the size of the donor compartment. An example is the contaminant-flow model:



The contaminant flow model can be written as a difference equation:

Contaminant Model as a Difference Equation:

$$x_{t+1} = x_t + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -(r_2 + r_3 + r_4) & 0 & 0 \\ r_3 & -(r_5 + r_6) & 0 \\ r_4 & r_5 & -r_7 \end{bmatrix} x_t$$

$$x_{t+1} = \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1-(r_2 + r_3 + r_4) & 0 & 0 \\ r_3 & 1-(r_5 + r_6) & 0 \\ r_4 & r_5 & 1-r_7 \end{bmatrix} x_t$$

$$x_{t+1} = F + Rx_t$$

$$equilibria: x^* = -(R-I)^{-1}F$$

System is stable if -1< |λ| <1 where λ are eigenvalues of R

In words, the next x is equal to the previous x plus inputs F + outputs and transfers among compartments:

$\mathbf{x}_{t+1} = \mathbf{I}\mathbf{x}_t + \mathbf{F} + (\mathbf{R} - \mathbf{I})\mathbf{x}_t = \mathbf{F} + \mathbf{R}\mathbf{x}_t$

where

$$R \equiv \begin{bmatrix} 1-(r_2+r_3+r_4) & 0 & 0 \\ r_3 & 1-(r_5+r_6) & 0 \\ r_4 & r_5 & 1-r_7 \end{bmatrix}$$

At equilibrium, $\mathbf{x}_{t+1} = \mathbf{x}_t$ so

$0 = \mathbf{F} + (\mathbf{R} - \mathbf{I})\mathbf{x}^*$

$-\mathbf{F} = (\mathbf{R} - \mathbf{I})\mathbf{x}^*$

$-\mathbf{F}\,(\mathbf{R} - \mathbf{I})^{-1} = \mathbf{x}^*$

For the whole system, the turnover rate at equilibrium is flux/standing stock or sum(**F**)/sum(**x***) and the residence time is standing stock/flux or sum(**x***)/sum(**F**). For an individual compartment i, if r_{ii} stands for the corresponding diagonal element of **R**, the turnover rate at equilibrium is $(r_{ii} - 1)x_i^*/x_i^* = r_{ii} - 1$. The residence time is the reciprocal of turnover rate, $1/(r_{ii} - 1)$.

The equilibrium is stable if $-1 < |\lambda| < 1$ where λ are eigenvalues of **R** and $|\lambda|$ is the magnitude of the eigenvalue. The return time of the entire system to steady state, following a perturbation, is proportional to $1/|\lambda_{max}|$ where λ_{max} is the largest eigenvalue of **R**. <u>Definition of Magnitude of an Eigenvalue:</u> If the eigenvalue is real, $|\lambda|$ is the same as the absolute value of eigenvalue. If the eigenvalue is complex, $|\lambda|$ is the length of the eigenvalue on the complex plane. Specifically, if λ is the complex number a + bi, $|\lambda| = \text{sqrt}(a^2 + b^2)$.